SEMICONVECTION AND OVERSHOOTING: SCHWARZSCHILD AND LEDOUX CRITERIA REVISITED

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ABSTRACT

We show that the Schwarzschild formalism, if accepted as the description of the structure of a semiconvective region, necessarily implies overshooting. The Ledoux formalism (if accepted) can be used without overshooting, provided a new relation for the mixing-length parameter α is satisfied. Should the latter not be satisfied, the Ledoux formalism would also require overshooting, that is, a nonlocal convection model.

Subject headings: convection — stars: interiors — turbulence

1. FORMULATION OF THE PROBLEM

In massive stars, there exists a *semiconvection zone*, a chemically inhomogeneous zone between the convective core and the radiative envelope (Schwarzschild & Härm 1958). The still unsolved problem is as follows: what are the values of $\nabla (= \partial \ln T / \partial \ln P)$ and $\nabla_\mu (= \partial \ln \mu / \partial \ln P)$ in such a zone? Here μ is the mean molecular weight of the stellar material. To discuss the problem, we adopt a formalism (Canuto 1999, hereafter C99) that can also be used to study concentration problems. We consider a system with two fluids, with density ρc and $(1 - c)\rho$, respectively, where c is the relative concentration. Each variable of the problem (velocity, temperature, and c) has mean parts U, T, and C and fluctuating or turbulent components u'', T'', and c''. The mean fields T and C are governed by the following equations:

$$\rho c_v \frac{\partial T}{\partial t} + \rho \left[\frac{\partial K}{\partial t} + \frac{\partial}{\partial z} F(K) \right] = -\frac{\partial}{\partial z} \rho c_p (J_r + J_h),$$

$$\frac{\partial C}{\partial t} = -\rho^{-1} \frac{\partial}{\partial z} \rho J_c. \tag{1}$$

Here K is the turbulent kinetic energy, $F(K)$ is the flux of K, and J_r, J_h, and J_c are the radiative, convective, and concentration fluxes, respectively. We have

$$J_r = \chi T H_p^{-1} \nabla, \quad J_h = \chi_h T H_p^{-1} (\nabla - \nabla_{\mathrm{ad}}), \quad J_c = -\chi_c \frac{\partial C}{\partial z}. \tag{2a}$$

Here $\chi = 4acT^3 / 3c_p \kappa \rho^2$ is the radiative diffusivity and χ_h and χ_c are the *turbulent diffusivities* for *heat and concentration*, respectively. The last two relations in equation (2a) are derived equations (160a) and (160b) of C99, with a slight change of notation. In terms of ∇_μ, $\partial C / \partial z = -\alpha_c^{-1} H_p^{-1} \nabla_\mu$, where $\alpha_c = (\partial \ln \rho / \partial C)_{P,T}$. The dynamic equation for K is

$$\frac{\partial K}{\partial t} + \frac{\partial}{\partial z} F(K) = P - \epsilon, \tag{2b}$$

while that for its dissipation ϵ is ($c_1 = 1.44$, $c_2 = 1.98$)

$$\frac{\partial \epsilon}{\partial t} + \frac{\partial}{\partial z} F(\epsilon) = \epsilon K^{-1} (c_1 P - c_2 \epsilon). \tag{2c}$$

The production P is defined as

$$P = gT^{-1} J_h - g\alpha_c J_c = gH_p^{-1} [\chi_h (\nabla - \nabla_{\mathrm{ad}}) - \chi_c \nabla_\mu]. \tag{2d}$$

A turbulence model is needed to express $F(K)$, the flux of ϵ, $F(\epsilon)$, and the diffusivities χ_h and χ_c. Once these variables are given, equations (1)–(2d) can be solved in the semiconvective region of a star. The solution yields the values of ∇ and ∇_μ.

2. SCHWARZSCHILD AND LEDOUX CRITERIA

Lacking such solution, Schwarzschild & Härm (1958) suggested the following criterion (the S criterion for brevity):

$$\nabla = \nabla_r, \quad \nabla = \nabla_{\mathrm{ad}}, \tag{3a}$$

while Ledoux (1947) and Sakashita & Hayashi (1959) suggested the alternative criterion (the L criterion; for simplicity, we treat an ideal gas):

$$\nabla = \nabla_r, \quad \nabla_\mu = \nabla - \nabla_{\mathrm{ad}}. \tag{3b}$$

We recall that these criteria were designed for hydrogen-rich stellar environments since, under hydrogen-free conditions, ∇_r does not necessarily depend on the chemical composition. The question that is still unresolved since 1958 is which of the two criteria is correct, if either. Testing the two criteria on purely astrophysical data has led to contrasting results. For example, Stothers & Chin (1992, 1994) concluded in favor of the second of equations (3b); Umezu (1998) has recently concluded that this equation gives unphysical results while equations (3a) do not. It seems that astrophysical tests alone are not sufficient. For a detailed, critical discussion of past models, see Merryfield (1995).

3. RECENT WORK

Several authors have proposed models for semiconvection that embrace both criteria in special limits, and thus specific stellar cases will naturally gravitate toward the criterion that is physically relevant. Although based on different premises, the models by Langer, El Eid, & Fricke (1985) and Grossman & Taam (1996) lead to essentially the same result:

$$\frac{\chi_c}{\chi} = \frac{1}{6} \alpha_{\mathrm{sc}} \frac{1}{R_\mu - 1}, \quad R_\mu \equiv \frac{\nabla_\mu}{\nabla - \nabla_{\mathrm{ad}}}. \tag{4}$$

These relations hold for dynamically stable situations, and thus, by definition, $R_\mu > 1$. In Langer et al. (1985), α_{sc} is an adjustable parameter to account for the uncertainties in translating a laminar growth rate into a nonlinear mixing. For the case of a
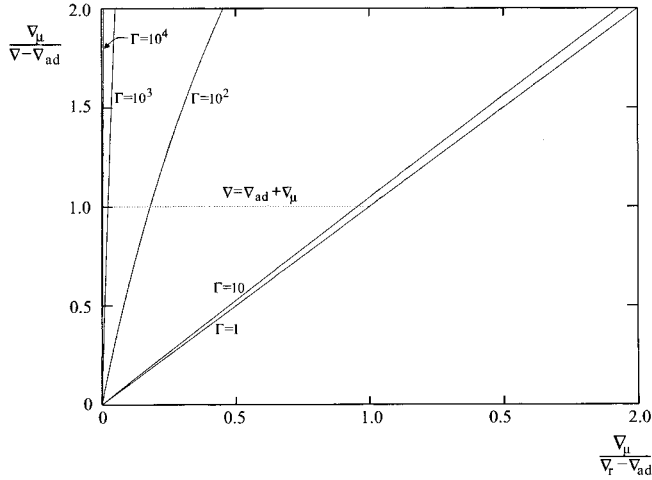
FIG. 1.—Solution of the flux conservation law in eqs. (5e), with the turbulent heat diffusivity given by eqs. (5a) and (5b) and the $P = \epsilon$ relation given by eqs. (5d). The curves are for $\Gamma = 10^{-1}$, 10^{-2}, 10^{-3}, etc., are indistinguishable from the $\Gamma = 1$ curve.

30 $M_\odot$ star, the resulting temperature profiles of Langer et al. (1985) are such that $(\nabla - \nabla_{ad})_{SCZ} \approx 10^4 (\nabla - \nabla_{ad})_{CZ}$, where CZ stands for the fully convective zone. A similar result was obtained by Grossman & Taam (1996). However, as the authors point out, their models suffer from adjustable parameters that cannot be determined from within the models.

4. PRESENT WORK

Recently (C99), a nonlocal and time-dependent dynamical one-point closure turbulence model was derived to treat semiconvection. It provides the dynamic equations for the turbulent variables of interest in the presence of the three external gradients of T, C, and U (e.g., differential rotation). The C99 model is, in principle, valid for arbitrary values of the radiative opacity χ (see after eqs. [2a]) and for arbitrary values of the Peclet number ($=\chi_h/\chi$, the ratio of turbulent to radiative heat diffusivity), which is an indication of the efficiency of the turbulent over radiative transport of heat. To obtain manageable results, only two dynamic, nonlocal, time-dependent equations were kept, those for K and ϵ (eqs. [2b] and [2c]). This allowed an analytic solution to be obtained for all the remaining turbulent variables. In particular, the expressions for $\chi_{h,c}$ were derived to be

$$\chi_h = \nu_T A_h, \quad \chi_c = \nu_T A_c, \quad \nu_T = \frac{28}{15}\frac{K^2}{\epsilon} \quad (5a)$$

and

$$A_{h,c} = A_{h,c}(\nabla_\mu, \nabla - \nabla_{ad}, \chi; K, \epsilon), \quad (5b)$$

where ν_T plays the role of a turbulent viscosity and the explicit form of A is given by equations (162a)–(162c) of C99.

5. LOCAL MODEL

In this Letter, we analyze semiconvection in the local, time-independent limit of equations (2b) and (2c), which become

$$P = \epsilon, \quad \epsilon = K^{3/2}\Lambda^{-1}, \quad (5c)$$

where Λ is a mixing length. Using equations (2d) and (5a), the

$P = \epsilon$ relation becomes

$$A_h(\nabla - \nabla_{ad}) - A_c\nabla_\mu = K/K_0, \quad K_0 = \frac{28}{15}g\Lambda^2 H_p^{-1}. \quad (5d)$$

Taking the stationary limit of the T-equation but not of the C-equation, equations (1) yield

$$\nabla + \frac{\chi_h}{\chi}(\nabla - \nabla_{ad}) = \nabla_r, \quad \frac{\chi_c}{\chi}\nabla_\mu = J_*, \quad (5e)$$

where

$$J_*(z, t)J_0 = J_t(z, t) + \tilde{\rho}\tilde{J}\rho^{-1} \quad (5f)$$

and

$$J_t(z, t) = -\rho^{-1}\int \rho(z')C_{,t}\, dz', \quad (5g)$$

with $J_0 = \chi\alpha_c^{-1}H_p^{-1}$, $\rho = \rho(z)$, and $C_{,t} = \partial C(z', t)/\partial t$. We have called $\tilde{\rho}\tilde{J}$ the constant of integration. The first of equations (5e) is the ordinary flux conservation law; the second relation in equations (5e) has been written so as to exhibit a similar structure: J_* plays the role of ∇_r, but since there is no analog of the radiative flux, the equivalent of the ∇ term is missing. The problem can thus be formulated as follows: we search for the two gradients ∇ and ∇_μ, which are solutions of equations (5e), provided ∇_r and J_* are given. However, since the turbulent diffusivities χ_h and χ_c depend on the turbulent kinetic energy (see eqs. [5a] and [5b]), one more equation is needed and that is equation (5d). We thus have three unknowns, ∇, ∇_μ, and K, and three coupled equations, equations (5d) and (5e). The dimensional variables of the problem combine to give rise to a dimensionless efficiency factor Γ given by

$$\Gamma = \frac{8\pi^2}{125}(\nabla_r - \nabla_{ad})^{1/2}\frac{\tau_{rad}}{\tau_{dyn}}, \quad \tau_{rad} = \Lambda^2\chi^{-1},$$

$$\tau_{dyn} = (H_p g^{-1})^{1/2}. \quad (6a)$$

The numerical solution of the problem, which entails only algebraic relations, is presented in Figures 1, 2, and 3 in the form

$$\frac{\nabla_\mu}{\nabla - \nabla_{ad}} \text{ versus } \frac{\nabla_\mu}{\nabla_r - \nabla_{ad}}, \quad \frac{\nabla_\mu}{\nabla - \nabla_{ad}} \text{ versus } J_*. \quad (6b)$$

In our treatment, no assumptions are made about ∇_r and/or J_*, whose values depend on the specific stellar case under consideration. Several conclusions ensue:

S criterion.—The condition $\nabla = \nabla_r = \nabla_{ad}$ can satisfy the first of equations (5e) but not equations (5d). The physical reason is clear. In equations (5d), the first term acts like a source, provided $\nabla - \nabla_{ad} > 0$, while ∇_μ acts like a sink. If the source vanishes because $\nabla \to \nabla_{ad}$, there is no way to balance the sink ∇_μ; there is no turbulent kinetic energy and thus no mixing. On the other hand, if we employ a *nonlocal model*, equation (2b) gives (instead of eqs. [5d])

$$A_h(\nabla - \nabla_{ad}) - A_c\nabla_\mu = K/K_0\left[1 + \Lambda K^{-3/2}\frac{\partial}{\partial z}F(K)\right]. \quad (6c)$$
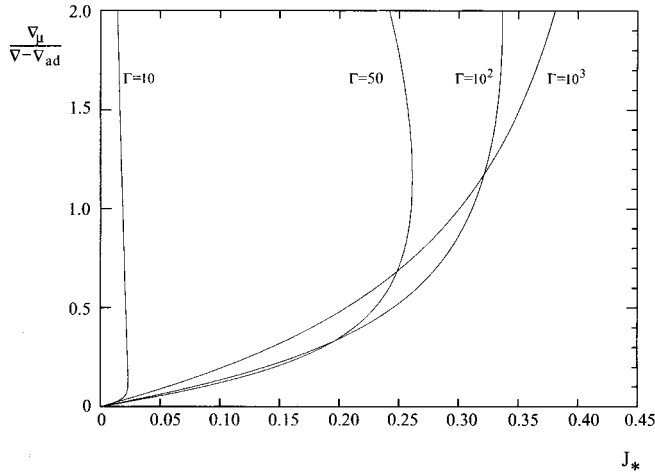
FIG. 2.—Some solutions of the concentration flux conservation law in eqs. (5e), with the turbulent diffusivities given by eqs. (5a) and (5b).
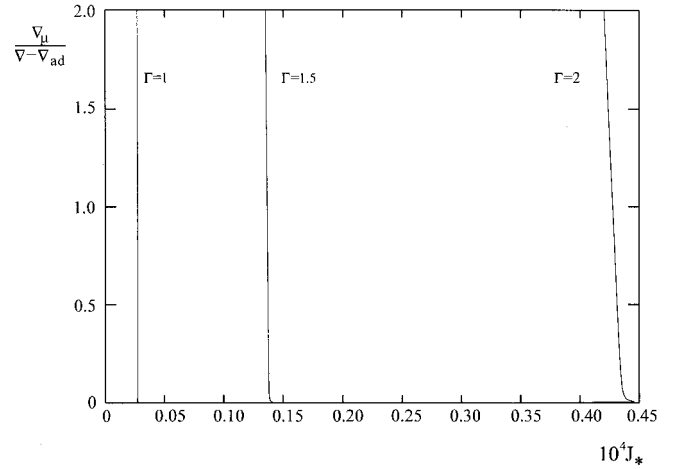

FIG. 3.—Expanded version of Fig. 2. For $\Gamma = 10^{-1}$ and $\Gamma = 10^{-2}$, the corresponding values of J_* are 3×10^{-10} and 3×10^{-14}, respectively.

In those regions where the divergence of the flux $F(K)$ is negative, nonlocality acts like a source of kinetic energy that, via diffusion, can counterbalance the sink represented by ∇_μ. Thus, in a local model, the S criterion $\nabla = \nabla_r = \nabla_{ad}$ *cannot be satisfied, while in a nonlocal model, it may be satisfied.*

L criterion.—We distinguish two possibilities. If we limit ourselves to the second of equations (3b), Figure 1 shows that it can be satisfied by many Γ for each of which there is a unique ∇_μ. Some solutions of the second of equations (5e) are shown in Figures 2 and 3. A variety of J_* is also allowed. On the other hand, if we take the L criterion to mean both relations in equations (3b), then Figure 1 implies that only $\Gamma \leq 1$ are allowed. For some Γ, this region is presented in Figure 3. From these results and the ones cited in the caption to Figure 3, we have found that the following relation holds:

$$J_* = 3 \times 10^{-6}\Gamma^4, \tag{6d}$$

which represents a constraint between the μ-gradient J_* and the convective efficiency Γ. There is another way of looking at equation (6d). In fact, we can change equation (6d) into a constraint on the value of α, $\Lambda = \alpha H_p$. Using equations (6a), equation (6d) gives

$$\alpha = 6.2(\nabla_r - \nabla_{ad})^{-1/4}J_*^{1/8}(\chi^2 g^{-1}H_p^{-3})^{1/4}. \tag{6e}$$

6. CONCLUSIONS

We have shown that the S criterion is not compatible with the absence of overshooting. Stellar structure calculations that use the S criterion but not overshooting are not internally consistent. The L criterion is compatible with no overshooting so long as the new relations in equation (6d) or equation (6e) are satisfied. Should they be found incompatible with independent evidence, the L criterion would also imply overshooting.

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